Separability of Low-Momentum Effective Nucleon-Nucleon Potential

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A realistic nucleon-nucleon potential is transformed into the low-momentum effective one (LMNN) by the Okubo theory. The separable potentials are converted from LMNN by the universal separable expansion method and the simple Legendre expansion. By the calculation of triton binding energies the separability for the convergence of these ranks is evaluated. There is a tendency that the lower momentum cutoff parameter Λ of LMNN gains good separability.

1. *Introduction* Unified theory is regarded as an *integration* of some independent Hilbert spaces, and effective (equivalent) theory or renormalization is a *differentiation* into physically interested space (P space) and the uninterested rest of one (Q space). Both theories are related strongly and they are main themes of our physics. The relation between unified theory and effective theory is explicitly explained by the Okubo effective theory.[1)] The theory is universally useful in many physics. For example, there is a good application of the Okubo theory to the recent chiral perturbation theory[2)] in the meson-nucleon systems. The original Lagrangian of the nucleon (N) and pion (π) fields generates the NN interaction. The bare NN interaction is connected with the πNN sector we need to sweep up the sector up to the πN threshold.

In the many-nucleon system of nuclear physics Suzuki and Okamoto have extended it to the useful scheme called the unitary-model-operator approach (UMOA).[3),4)] The UMOA is an approach aimed at a many-body system by considering the effective interactions in nuclear medium, which is determined by solving the decoupling

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equation between the model space and its complement space. When the effective theory in the same sense of Okubo theory and UMOA is applied to a two-body system, it generates the low-momentum nucleon-nucleon (LMNN) potential by defining a model space (P space) and its complement (Q space). Bogner *et al.*[5),6)] suggested that their LMNN made by G-matrix scheme is useful to apply to many-body systems.

In order to look into the accuracy of LMNN one needs to calculate the triton and the alpha-particle binding energies where the Faddeev equation or the Yakubovsky equation gives the exact solutions to the given potential. It was concluded in ref.[7)] that in the case of the realistic NN forces, g.e., Nijm-I[8)] or CD-Bonn[9)] potentials, the recommended truncation Λ is, at least, larger than 5 fm^{-1} to reproduce the exact values of the binding energies in these systems. The calculation of the ground state energy using the LMNN for the cutoff parameter $\Lambda \approx 2$ fm^{-1} yields considerably more attractive result than the exact value. Variational principle is in possession a repulsion property which its absolute value of the binding energy is less than the true one. There could be an accidental cancellation between the attraction caused from the short cutoff parameter Λ and the repulsion from the variational principle.

Besides the above discussion, we would like to investigate another property of the LMNN interaction. When the Hilbert space is in general truncated into the small P space, the structure of the bases is expected to be much simple and regular. The NN interaction is expanded in a separable form, which strongly reduces the numerical cost of a relatively heavy calculation in the few-body systems.[10),11)] Namely, we try to restrict the degrees of freedom for the continuum variables in the integral equations by introducing the separable potential. In the case of a three-nucleon system the accuracy of the calculation is examined by some benchmarks. The separable potential has a rank of the form factors which describe the behavior of the potential. The more precise and the lower ranks, the better. The simplicity of the P space is considered to be reflected in the convergence of the rank or the separability. In the application to the few-body calculation it is interesting whether the LMNN potential has a merit of the separability. We expect that the LMNN potential has good separability and it will reduce the numerical cost of calculations in our physics.

In the next section we introduce two kinds of the separable expansion. The triton binding energies are calculated by using these finite rank separable potentials in section 3. We would like to look into the convergence of the rank or the separability in order to show how the Hilbert space is effectively simplified. Discussion and outlook are given in section 4.

2. *Simple separable expansion and the universal separable expansion* In Ref.[7)] the LMNN interaction was obtained by using two kinds of methods. It was numerically confirmed that both methods proposed by Glöckle-Epelbaum[12)] and by Suzuki-Okamoto[3),4)] lead to the same LMNN interaction. The LMNN potential is fulfilled by the following Lippmann-Schwinger equation at energy E;

$$T(p,p';E) = V(p,p') + \int_0^{\Lambda} V(p,p'')G_0(p'',E)T(p'',p';E)p''^2 dp'', \tag{1}$$

where T, V, G_0 and Λ are the transition matrix (t-matrix), LMNN potential, Green's function of free two particles and the cutoff parameter of the low momentum, respectively.

Now, the momentum variable p in the integral is replaced by the other variable x defined as

$$p = \frac{x+1}{2}\mathit{\Lambda}, \tag{2}$$

and the potential and the t-matrix are expanded by the separable forms

$$V(p,p')\ pp' \approx V^{sep}(p,p')\ pp' \equiv \sum_{i,j=1}^{n} g_i(x)\lambda_{i,j}g_j(x') \tag{3}$$

and

$$T(p,p';E)\ pp' \approx \sum_{i,j}^{n} g_i(x)\tau_{i,j}(E)g_j(x'), \tag{4}$$

where g, λ are the form factor and the coupling constant, and we rewrite Eq. (1);

$$\tau_{i,j}(E) = \lambda_{i,j} + \sum_{k,l}^{n} \lambda_{i,k}I_{k,l}\tau_{l,j}(E) \tag{5}$$

with

$$I_{k,l} \equiv \frac{\mathit{\Lambda}}{2}\int_{-1}^{1} g_k(x)G_0(p(x))g_l(x)dx. \tag{6}$$

Equation (5) is algebraically solved by the matrix inversion method. The number n means the rank of the separable expansion.

The interval of integration is finite and the potential has no singularity, therefore, we expect the LMNN potential is easily expanded by the simple polynomials. The Legendre function $P_i(x)$ may be naturally chosen for such polynomials;

$$g_i(x) = P_i(x) \tag{7}$$

and

$$\lambda_{i,j} = \frac{(2i+1)(2j+1)}{4}\int_{-1}^{1}\int_{-1}^{1} V(p,p')pp'P_i(x)P_j(x')dxdx'. \tag{8}$$

The expansion is nothing new, even in the infinite boundary condition, but the Hanover group has succeeded[13)] to perform the accurate Faddeev calculations for the proton-deuteron scattering by using the Chebyshev polynomials. In section 3 we call it the simple separable expansion (SSE).

On the other hand, one of the well-developed separable expansion scheme has been introduced.[10)] The new form factor g_i is defined as

$$g_i(p) =< p|g_i >\equiv< p|V|P_i >= \int_{-1}^{1} V(p,p'(x'))P_i(x')dx' \tag{9}$$

and

$$V^{USE}(p,p')\ pp' \equiv < p|g > \lambda < g|p' > = \sum_{i,j}^{n} g_i(p)\lambda_{i,j}^{USE} g_j(p') \tag{10}$$

with

$$\left[\lambda_{i,j}^{USE}\right]^{-1} = \int_{-1}^{1}\int_{-1}^{1} V(p,p')pp'P_i(x)P_j(x')dxdx', \tag{11}$$

where $[\ ]^{-1}$ in Eq.(11) means the matrix inversion. The polynomials (Legendre functions in this case) are only required for the linear independence, while Eq. (7) of the SSE needs orthonormality. Therefore, one understands this expansion is a more general method. We suppose to call it the universal separable expansion (USE). In the case of the Faddeev calculation for *nd* scattering the high convergence was emphasized,[10] but we would like to investigate the separability of the LMNN interaction by using the SSE and the USE.

3. *The calculation of the triton binding energies using the USE and the SSE* The dependence of the accuracy of the LMNN potential on the cutoff momentum Λ had already been investigated.[7),14] We are now interested in how the separability will be developed by changing Λ. For example, we employ the CD-Bonn potential[9] which is well-known as one of the modern and precise potentials.

The triton binding energies are calculated by the USE and SSE for various values of cutoff Λ as well as Ref.[7] For the sake of simplicity the calculation is performed only 5-channel coupled Faddeev equation. Namely, the potential is used only for $^1S_0$ and $^3S_1$-$^3D_1$ states. The results are shown in Table I. In the second line the exact values of the finite values of Λ, calculated without the separable approximation, are shown, while the true value ($\Lambda = \infty$) is -8.312 MeV.

Table I. The triton binding energies. The energy is in MeV. The exact values under the cutoff Λ are in the second line. The "SSE" and "USE" denote the simple separable expansion and the universal separable expansion, respectively

	Λ= 3 fm^{-1}		Λ= 5 fm^{-1}		Λ= 10 fm^{-1}		Λ= 20 fm^{-1}	
	-8.532		-8.355		-8.329		-8.322	
rank n	SSE	USE	SSE	USE	SSE	USE	SSE	USE
20	**-8.532**	**-8.532**	**-8.354**	**-8.355**	-8.319	**-8.329**	-8.317	-8.320
18	**-8.532**	**-8.532**	-8.353	**-8.355**	-8.319	-8.327	-8.308	-8.319
16	**-8.532**	**-8.532**	-8.349	**-8.354**	-8.253	-8.382	-7.762	-7.973
14	-8.531	**-8.532**	-8.346	**-8.354**	-7.798	-8.305	-6.833	-7.196
12	-8.526	**-8.532**	-8.328	-8.351	-6.605	-8.203	-5.733	-8.089
10	-8.523	**-8.532**	-7.963	-8.311	-5.645	-7.958		
8	-8.326	-8.509	-6.730	-8.125				
6	-7.182	-8.014	-6.417	-7.741				

The thickly indicating numbers in Table I perfectly agree with the exact ones for each Λ= 3, 5 and 10 fm^{-1}. Comparing the SSE and the USE one sees that the USE has a good convergence property because in the low rank steps the USE leads to the corresponding exact numbers. The effective potential has a tendency of the better separability in the lower-rank separable form. The lower value of Λ gives the higher separability of the LMNN interaction.

4. *Discussion and outlook* We calculated the triton binding energies by employing the LMNN CD-Bonn potential with some cutoff parameters Λ in the unitary-transformation method of Okubo theory. There is a tendency that the lower cutoff parameter we take, the better separability. The result is getting to be close to the exact value calculated with the high-rank separable potential, and the obtained exact value depends on the cutoff parameter.

It is well known that the binding energy consists of the positive expectation value of the kinetic part ($\sim$ 50MeV) and the negative expectation one from the potential ($\sim$ –60MeV). The difference of 43keV from the true value in the case of $\Lambda = 5\text{fm}^{-1}$ is interpreted only 0.1% error within the potential expectation value. The cross section in the scattering process is evaluated out of the potential expectation value, therefore, such a 0.1% error could be invisibly small. The Faddeev three-body scattering calculation is precisely obtained without the separable expansion,[15)–17)] but, for the case of the four-body scattering one still needs a lot of efforts to obtain the precise and stable solutions for it. The separable scheme reduces the size of memory and the cpu time in computer. The integral technique of the contour deformation requires the analytical property (analyticity) of the form factor function to avoid the logarithmic singularities which arise from the two-body t-matrix and the Green's function in momentum space. The schemes solving the Faddeev equation with the separable expansion method have usually been employing this contour-deformation technique.[19)]

The LMNN potential apparently has no analyticity of the form factor, and therefore it is not easy to apply the contour-deformation technique to the few-body scattering problem. Recently the complex energy method (CEM) is introduced.[18)] The CEM enables us the calculation without the analyticity of the form factor because the idea of the CEM makes good use of the analytic continuation of energy. The solution is obtained only by using the complex analytic continuation from sample solutions of the complex energies near the on-energy shell. The idea of the complex energy was introduced in order to avoid the dangerous singularities.

We have a plan that the finite-rank precise separable potential will be made out of the LMNN interaction, which is guaranteed by the good separability as shown in the present work. The calculation by using the USE potential not only for the three-body scattering problem but also for the four-body scattering the relevant scattering problems will be made precisely by the CEM.

The numerical calculations were performed mainly on a IBM RS/6000SP (Reserch Center of Nuclear Physics, Osaka University) in Japan and partly on a Hitachi SR8000 (Leibnitz-Rechenzentrum für die München Hochschule) and a Cray SV1 (NIC, Jülich) in Germany.

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